

An internal topological characterization of the subspaces of Eberlein compacts and related compacts – I

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Abstract

We obtain an internal topological characterization of the subspaces of Eberlein compacts (respectively, Corson compacts, strong Eberlein compacts, uniform Eberlein compacts, n -uniform Eberlein compacts).

1 Introduction

In 1977, A. V. Arhangel'skiĭ proved that every metrizable space has a compactification which is an Eberlein compact (see [5, Theorem 14]) (all necessary definitions are given in the next section). In 1982 (in a private communication), he posed the question: “When does a space X have a compactification cX which is an Eberlein compact?”. Since the closed subspaces of Eberlein compacts are Eberlein compacts, this question is equivalent to the following problem: find an internal characterization of the subspaces of Eberlein compacts (note that H. P. Rosenthal [42] gave an internal characterization of Eberlein compacts). In this paper we obtain such a characterization and, moreover, we characterize internally the subspaces of Corson compacts, of uniform Eberlein compacts, of n -uniform Eberlein compacts and of strong Eberlein

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compacts (see Theorems 3.6 and 3.12); all these results, with the exception of that about the subspaces of n -uniform Eberlein compacts, were announced (without any proofs) in my paper [16]. The cited above theorem of A. V. Arhangel'skiĭ, as well as its recent generalizations obtained by T. Banach and A. Leiderman [10, Theorem 1] and B. A. Pasynkov [33, Corollary 5], follow immediately from our results; the same is true for two assertions of J. Lindenstrauss [29, Propositions 3.1 and 3.2]. The paper contains also some results which were not announced in [16]; the main of them are: (1) the new characterizations of Eberlein compacts, Corson compacts, uniform Eberlein compacts, n -uniform Eberlein compacts and strong Eberlein compacts (see Theorem 3.4 and Fact 3.11), (2) the equalities $hl(X) = c(X) = hc(X)$ for subspaces X of Eberlein compacts (see Corollary 3.25), (3) the characterization of the spaces which are co-absolute with (zero-dimensional) Eberlein compacts (see Theorem 4.7), (4) a new proof of the famous Ponomarev Theorem [38] giving a solution to the Birkhoff Problem 72 [13], and (5) a discussion of the question whether each Tychonoff (or normal T_1) space with a uniform base (in the sense of P. S. Alexandrov [1]) is homeomorphic to a subspace of some Eberlein compact (see Remark 3.26). The notion of an *almost subbase* (introduced in [16]) plays a central role in all these results and in the whole paper.

Not all of the assertions announced in the paper [16] are proved here; the proofs of the remaining portion of them will be given in the second part of this paper.

We now fix the notation.

If A is a set, we denote by $|A|$ its cardinality. If (X, τ) is a topological space and M is a subset of X , we denote by $\text{cl}_{(X, \tau)}(M)$ (or simply by $\text{cl}(M)$ or $\text{cl}_X(M)$) the closure of M in (X, τ) and by $\text{int}_{(X, \tau)}(M)$ (or briefly by $\text{int}(M)$ or $\text{int}_X(M)$) the interior of M in (X, τ) . The Alexandroff compactification of a locally compact Hausdorff non-compact space X is denoted by αX , the set of all positive natural numbers – by \mathbb{N} , the real line (with its natural topology) – by \mathbb{R} , and the subspaces $[0, 1]$ and $\{0, 1\}$ of \mathbb{R} – by \mathbb{I} and \mathbb{D} , respectively. As usual, $\omega = \mathbb{N} \cup \{0\}$.

Let X be a dense subspace of a space Y and $U \subseteq X$. The *extension of U in Y* , denoted by $Ex_Y U$, is the set $Y \setminus \text{cl}_Y(X \setminus U)$; recall that if U is open in X then $Ex_Y U$ is the greatest open subset of Y whose trace on X is U .

If X is a topological space and $f : X \rightarrow \mathbb{I}$ is a continuous function, then we write, as usual, $Z(f) = f^{-1}(0)$ (*the zero-set of f*) and $\text{coz}(f) = X \setminus Z(f)$ (*the cozero-set of f*). We set $Z(X) = \{Z(f) \mid f : X \rightarrow \mathbb{I} \text{ is a continuous function}\}$ and $\text{Coz}(X) = \{\text{coz}(f) \mid f : X \rightarrow \mathbb{I} \text{ is a continuous function}\}$.

We denote by \mathbb{Q}_1 the set of all positive rational numbers less than 1.

Recall that a subset M of a topological space X is called *regular closed* (respectively, *regular open*) if $M = \text{cl}(\text{int}(M))$ (respectively, $M = \text{int}(\text{cl}(M))$); we denote by $RC(X)$ (respectively, $RO(X)$; $CO(X)$) the collection of all regular closed (respectively, regular open; clopen (= closed and open)) subsets of X . Recall also that $RC(X)$ becomes a complete Boolean algebra $(RC(X), 0, 1, \vee, \wedge, *)$ under the following operations: $1 = X$, $0 = \emptyset$, $F^* = \text{cl}(X \setminus F)$, $F \vee G = F \cup G$, $F \wedge G = \text{cl}(\text{int}(F \cap G))$. Also, $RO(X)$ becomes a complete Boolean algebra $(RO(X), 0, 1, \vee, \wedge, *)$ under the

following operations: $U \vee V = \text{int}(\text{cl}(U \cup V))$, $U \wedge V = U \cap V$, $U^* = \text{int}(X \setminus U)$, $0 = \emptyset$, $1 = X$. These two Boolean algebras are isomorphic.

The operation “complement” in Boolean algebras will be denoted by “*”.

We denote by $S : \mathbf{Bool} \rightarrow \mathbf{Stone}$ the Stone duality functor between the category **Bool** of Boolean algebras and Boolean homomorphisms and the category **Stone** of compact zero-dimensional Hausdorff spaces and continuous maps (see, e.g. [28]).

The class of all objects of a category \mathcal{C} is denoted by $|\mathcal{C}|$.

All notions and notation which are not explained here can be found in [21, 45, 28].

By a “space” we will mean a “topological T_0 -space”.

2 Preliminaries

Let Γ be an index set and let \mathbb{I}^Γ (respectively, \mathbb{D}^Γ) be the Cartesian product of $|\Gamma|$ copies of \mathbb{I} (respectively, \mathbb{D}) with the Tychonoff topology. We set

$$\Sigma(\mathbb{I}, \Gamma) = \{x \in \mathbb{I}^\Gamma \mid |\{\gamma \in \Gamma \mid x(\gamma) > 0\}| \leq \aleph_0\},$$

$$\Sigma_*(\mathbb{I}, \Gamma) = \{x \in \mathbb{I}^\Gamma \mid (\forall \varepsilon > 0)(|\{\gamma \in \Gamma \mid x(\gamma) \geq \varepsilon\}| < \aleph_0)\},$$

$$\sigma(\mathbb{D}, \Gamma) = \{x \in \mathbb{D}^\Gamma \mid |\{\gamma \in \Gamma \mid x(\gamma) = 1\}| < \aleph_0\},$$

and the topology on these subsets of \mathbb{I}^Γ is the subspace topology. Obviously, $\sigma(\mathbb{D}, \Gamma) \subseteq \Sigma_*(\mathbb{I}, \Gamma) \subseteq \Sigma(\mathbb{I}, \Gamma)$.

Let \mathcal{U} be a family of subsets of a set X . Recall that, for $n \in \omega$, the *order* of the family \mathcal{U} is $\leq n$ if any $n+2$ members of the family \mathcal{U} have an empty intersection (i.e., each $x \in X$ is in at most $n+1$ members of \mathcal{U}); the order of \mathcal{U} is *infinite* if there is no $n \in \omega$ such that the order of \mathcal{U} is $\leq n$. Let $n \in \mathbb{N}$; the family \mathcal{U} is called *boundedly point-finite* (respectively, *n-boundedly point-finite* ([23])) if it has a finite order (respectively, is of order $\leq n-1$). If $m \in \mathbb{N}$ and the family \mathcal{U} is a union of countably many families \mathcal{U}_n , where $n \in \mathbb{N}$, we will say that \mathcal{U} is a *σ -point-finite* (respectively, *σ -boundedly point-finite*; *σ -m-boundedly point-finite*) family if all families \mathcal{U}_n are point-finite (respectively, boundedly point-finite; *m-boundedly point-finite*). The family \mathcal{U} is said to be *T_0 -separating* if, whenever $x \neq y$ are in X , then there exists $U \in \mathcal{U}$ such that $|U \cap \{x, y\}| = 1$; in this case we will also say that the family \mathcal{U} *T_0 -separates* X . Finally, when X is a topological space, the family \mathcal{U} is said to be *F-separating* (see [30]) if, whenever $x \neq y$ are in X , then there is some $U \in \mathcal{U}$ such that $x \in U$ and $y \notin \text{cl}_X(U)$, or vice versa.

A compact Hausdorff space is called an *Eberlein compact* (briefly, EC), if it is homeomorphic to a weakly compact (i.e., compact in the weak topology) subset of a Banach space ([29]). D. Amir and J. Lindenstrauss proved that a compact space is an Eberlein compact if and only if it can be embedded in $\Sigma_*(\mathbb{I}, \Gamma)$ for some index set Γ (see [3, Theorem 1]). An internal characterization of Eberlein compacts was given by H. P. Rosenhthal [42].

Theorem 2.1 ([42]) *A compact Hausdorff space X is an Eberlein compact if and only if it has a σ -point-finite T_0 -separating collection of cozero-sets.*

A compact Hausdorff space that is homeomorphic to a weakly compact subset of a Hilbert space is called a *uniform Eberlein compact* (briefly, UEC); this class of spaces was introduced by Benyamini and Starbird in [12]. Let us recall the following result:

Theorem 2.2 ([11]) *Let X be a compact Hausdorff space. Then X is a uniform Eberlein compact iff X has a σ -boundedly point-finite, T_0 -separating family of cozero sets.*

Compact subspaces of $\Sigma(\mathbb{I}, \Gamma)$ were called *Corson compacts* (briefly, CC) by E. Michael and M.E. Rudin [30]. The internal characterization of Corson compacts is given below:

Theorem 2.3 ([30]) *Let X be a compact Hausdorff space. Then X is a Corson compact iff X has a point-countable, T_0 -separating family of cozero sets.*

A compact space X is called a *strong Eberlein compact* (briefly, SEC) ([43, 11, 42]) if it can be embedded in $\sigma(\mathbb{D}, \Gamma)$ for some set Γ . Equivalently, a compact space is a SEC iff it has a point-finite, T_0 -separating family of clopen sets. K. Alster [2] proved that a space is a SEC iff it is a scattered EC.

We will need the following fundamental result:

Theorem 2.4 ([11, 26, 43, 30]) *The classes of EC's, UEC's, SEC's and CC's are closed under continuous images.*

Definition 2.5 ([16, 18]) Let X be a space and V a subset of it. If there exists a collection $U(V) = \{U_n(V) \mid n \in \mathbb{N}\}$ such that $V = \bigcup \{U_n(V) \mid n \in \mathbb{N}\}$ and $U_n(V) \subseteq U_{n+1}(V)$, $U_{2n-1}(V) \in Z(X)$, $U_{2n}(V) \in \text{Coz}(X)$ for every $n \in \mathbb{N}$, then we say that the set V is *U-representable* and the collection $U(V)$ is a *U-representation of V* . If α is a family of subsets of X and, for every $V \in \alpha$, $U(V)$ is a U-representation of V , then the family $U(\alpha) = \{U(V) \mid V \in \alpha\}$ is called a *U-representation of α* .

The next lemma is standard:

Lemma 2.6 ([18]) *Let X be a space and $U(V) = \{U_n(V) \mid n \in \mathbb{N}\}$ be a U-representation of a subset V of X . Then there exists a continuous function $f : X \rightarrow \mathbb{I}$ such that $V = \text{coz}(f)$ and $f^{-1}([\frac{1}{2^{n-1}}, 1]) = U_{2n-1}(V)$, for every $n \in \mathbb{N}$.*

The definition given below was inspired by the results of B. Efimov and G. Čertanov [20]:

Definition 2.7 ([16]) A family α of subsets of a space X is said to be an *almost subbase* of X if there exists a U-representation $U(\alpha)$ of α such that the family $\alpha \cup \{X \setminus U_{2n-1}(V) \mid V \in \alpha, n \in \mathbb{N}\}$ is a subbase of X .

Remark 2.8 Every almost subbase α of a space X consists of cozero-sets because a subset of X is U-representable iff it is a cozero-set. Obviously, a U-representable set V can have many different U-representations.

It is easy to see that a space has an almost subbase iff it is completely regular (see [16, 18]).

Let us note that in [18] the notion of almost subbase is introduced in a little bit different manner, namely, there is there an additional requirement that almost subbases have to be T_0 -separating families. In [18] we work with arbitrary topological spaces. In this paper, as in [16], we work with T_0 -spaces only and this allows us to remove the extra requirement from [18]. Indeed, the condition that an almost subbase α of an arbitrary topological space X is a T_0 -separating family is used in [18] only for showing that the family $\mathcal{F}_\alpha = \{f_V \mid V \in \alpha\}$, where f_V is the function constructed in Lemma 2.6 on the base of the given U-representation $U(V)$ of V , separates points; if X is a T_0 -space, however, it is easy to see that the family \mathcal{F}_α separates points even when α is not required to be a T_0 -separating family (indeed, if $x \neq y$ are in X and there is no $V \in \alpha$ such that $|V \cap \{x, y\}| = 1$ then there is some $W \in \alpha$ and some $n \in \mathbb{N}$ such that $|(X \setminus U_{2n-1}(W)) \cap \{x, y\}| = 1$; then $f_W(x) \neq f_W(y)$). Also, arguing in a similar way, we obtain that if α is an almost subbase of a T_0 -space then the family

$$\alpha' = \alpha \cup \{f_V^{-1}((r, 1]) \mid V \in \alpha, r \in \mathbb{Q}_1\}$$

(where the functions f_V are constructed as above) is a T_0 -separating family (it is even an F-separating family) and an almost subbase. Moreover, when α is a σ -point-finite (respectively, σ -locally finite; point-countable) family then α' has the same property.

The next proposition (proved in [18]) generalizes the Nagata-Smirnov metrization theorem:

Proposition 2.9 ([16, 18]) *A space X is metrizable iff it has a σ -locally finite almost subbase.*

In [18], using Proposition 2.9, the following theorem was proved as well:

Theorem 2.10 ([16, 18]) *Every Baire subspace of $\Sigma_*(\mathbb{I}, \Gamma)$ contains a dense G_δ metrizable subspace.*

Corollary 2.11 ([11, 31]) *Every EC has a dense G_δ metrizable subspace.*

We will also need the following simple lemma:

Lemma 2.12 ([20]) *A continuous bijection $f : X \longrightarrow Y$ is a homeomorphism if and only if there exists a subbase \mathcal{B} of X such that $f(U)$ is open in Y for every $U \in \mathcal{B}$.*

Let us recall the following results and definitions from [18]:

Proposition 2.13 ([18]) *Let Y be a subspace of a space X and let α be an almost subbase of X . Then $\alpha \cap Y = \{V \cap Y \mid V \in \alpha\}$ is an almost subbase of Y .*

Lemma 2.14 ([18]) *Let X be a space and α be a T_0 -separating family of cozero-subsets of X . Let, for every $U \in \alpha$, a family $\alpha_U = \{U_i \in \text{Coz}(X) \mid i \in \mathbb{N}\}$ is given, such that $U = \bigcup \{U_i \mid i \in \mathbb{N}\}$ and $U_i \subseteq \text{cl}(U_i) \subseteq U_{i+1}$ for every $i \in \mathbb{N}$. Then $\alpha^* = \bigcup \{\alpha_U \mid U \in \alpha\}$ is an F -separating family of cozero-subsets of X .*

Definition 2.15 ([18, 19]) A family α of cozero-subsets of a space X is said to be a *uniform almost subbase* of X if, for every U -representation $U(\alpha)$ of α , the family $\alpha \cup \{X \setminus U_{2n-1}(V) \mid V \in \alpha, n \in \mathbb{N}\}$ is a subbase of X .

Lemma 2.16 ([18]) *If X is compact then every F -separating family α of cozero-subsets of X is a uniform almost subbase of X .*

Theorem 2.17 ([18, 19]) *A compact Hausdorff space X is an Eberlein compact iff X has a σ -point-finite (uniform) almost subbase.*

The next notion was introduced recently by B. A. Pasynkov [33]:

Definition 2.18 ([33]) Let $n \in \omega$. A compact Hausdorff space X is called *n -uniform Eberlein* (briefly, *n -UEC*) if X has a T_0 -separating family $\gamma = \bigcup_{i \in \mathbb{N}} \gamma_i$ of cozero sets such that the order of all families γ_i , for $i \in \mathbb{N}$, is $\leq n$ (i.e., in our terms, if X has a T_0 -separating, σ -($n+1$)-boundedly point-finite family of cozero-sets).

B. A. Pasynkov announced in [33, Final remark] that for any $n \in \mathbb{N}$, there exists an n -UEC that is not an $(n-1)$ -UEC.

3 On Eberlein spaces and related spaces

Definition 3.1 ([34, 22, 16]) A family α of subsets of a set X is called *strongly point-finite* (respectively, *strongly point-countable*) if every subfamily μ of α with $|\mu| = \aleph_0$ (respectively, $|\mu| = \aleph_1$) contains a finite subfamily μ' with empty intersection.

A family of subsets of a set X is called *σ -strongly point-finite* if it is a union of countably many strongly point-finite families.

Remark 3.2 Obviously, a family α of subsets of a set X is strongly point-finite iff it contains no infinite subfamily with finite intersection property. In this form, the notion of “strongly point-finite family” is introduced independently in [34] and [22] (in [34] there is no name for such families and in [35] they are called “b-families”); moreover, in [34], this notion is attributed to A. V. Arhangel’skiĭ. Of course, I was unaware of these facts when I introduced this notion in [16].

Clearly, a family α of subsets of a set X is strongly point-countable iff it contains no uncountable subfamily with finite intersection property. In this form, the notion of “strongly point-countable family” is introduced in [34] (as “condition B1”) but it is not studied there.

Evidently, every strongly point-finite (respectively, strongly point-countable) family is a point-finite (respectively, point-countable) family; the converse is not true.

Lemma 3.3 *Let X be a dense subspace of a space Y , $n \in \omega$ and γ be a strongly point-finite (respectively, boundedly point-finite, n -boundedly point-finite, strongly point-countable) family of open subsets of X . Then the family*

$$Ex_Y \gamma = \{Ex_Y U \mid U \in \gamma\}$$

is a strongly point-finite (respectively, boundedly point-finite, n -boundedly point-finite, strongly point-countable) family of open subsets of Y .

Proof. Let γ be a strongly point-finite family of open subsets of X , $\mu \subseteq Ex_Y \gamma$ and $|\mu| = \aleph_0$. Let $\mu_X = \{V \cap X \mid V \in \mu\}$. Then $|\mu_X| = \aleph_0$ and $\mu_X \subseteq \gamma$. Hence, there exists a finite subfamily μ'_X of μ_X with empty intersection. Since the operator Ex_Y preserves finite intersections, we get that $\mu' = Ex_Y \mu'_X$ is a finite subfamily of μ with empty intersection. Therefore, $Ex_Y \gamma$ is a strongly point-finite family of open subsets of Y .

The proof of the other three cases is analogous. □

In the proof of the case of Eberlein compacts of the next theorem, the constructions of the family α is taken from the Michael-Rudin proof of Rosenthal’s theorem (see [30, Theorem 1.4]).

Theorem 3.4 *Let $n \in \omega$ and X be a compact Hausdorff space X . Then X is an EC (respectively, CC; UEC; n -UEC) iff X has a σ -strongly point-finite (uniform) almost subbase (respectively, strongly point-countable (uniform) almost subbase; σ -boundedly point-finite (uniform) almost subbase; σ -($n + 1$)-boundedly point-finite (uniform) almost subbase).*

Proof. We start with the proof for Eberlein compacts.

(\Rightarrow) Let X be an EC. By the Amir and Lindenstrauss theorem, we can regard X as a subspace of $\Sigma_*(\mathbb{I}, \Gamma)$ for some index set Γ . Now, for every $\gamma \in \Gamma$ and every $r \in \mathbb{Q}_1$, we set

$$V_{r,\gamma} = \{x \in X \mid x(\gamma) > r\}, \quad \alpha_r = \{V_{r,\gamma} \mid \gamma \in \Gamma\}, \quad \alpha = \bigcup \{\alpha_r \mid r \in \mathbb{Q}_1\}.$$

We will prove that, for every $r \in \mathbb{Q}_1$, α_r is a strongly point-finite family.

Let $r \in \mathbb{Q}_1$ and $\mu = \{V_{r,\gamma_n} \mid n \in \mathbb{N}\}$, where $\gamma_n \neq \gamma_m$ for $n, m \in \mathbb{N}$ with $n \neq m$, be an infinite countable subfamily of α_r consisting of non-empty sets. Set $\bar{\mu} = \{\text{cl}_X(V_{r,\gamma_n}) \mid n \in \mathbb{N}\}$. Then the family $\bar{\mu}$ has empty intersection. Indeed, since $\text{cl}_X(V_{r,\gamma_n}) \subseteq \{y \in X \mid y(\gamma_n) \geq r\}$ for every $n \in \mathbb{N}$, if some point $x = (x(\gamma))_{\gamma \in \Gamma}$ of X belongs to every element of $\bar{\mu}$, then $x(\gamma_n) \geq r$ for every $n \in \mathbb{N}$ and this is a contradiction because $r > 0$ and $X \subseteq \Sigma_*(\mathbb{I}, \Gamma)$. So, $\bigcap \bar{\mu} = \emptyset$. Since X is compact, there exists a finite subfamily $\bar{\mu}_0 = \{\text{cl}_X(V_{r,\gamma_{n_i}}) \mid i = 1, \dots, k\}$ (where k is from \mathbb{N}) of $\bar{\mu}$ with empty intersection. Then the family $\mu_0 = \{V_{r,\gamma_{n_i}} \mid i = 1, \dots, k\}$ is a finite subfamily of μ with empty intersection. Therefore, α_r is a strongly point-finite family.

It is easy to see that α is an F-separating family. Then, by Lemma 2.16, α is a uniform almost subbase and, thus, it is an almost subbase.

(\Leftarrow) Let α be a σ -strongly point-finite almost subbase of X . Then the family α' constructed in Remark 2.8 is a T_0 -separating, σ -strongly point-finite almost subbase of X . Since every strongly point-finite family is point-finite, our assertion follows from Theorem 2.1.

We proceed with the proof for Corson compacts:

(\Rightarrow) Let X be a CC. Then we can regard X as a subspace of $\Sigma(\mathbb{I}, \Gamma)$ for some index set Γ . For every $\gamma \in \Gamma$ and every $r \in \mathbb{Q}_1$, we define the set $V_{r,\gamma}$ as above, and we set $\alpha = \{V_{r,\gamma} \mid \gamma \in \Gamma, r \in \mathbb{Q}_1\}$. We will prove that α is a strongly point-countable family. Let $\mu = \{V_{r_\xi, \gamma_\xi} \mid \xi < \omega_1\}$, where $(r_\xi, \gamma_\xi) \neq (r_\eta, \gamma_\eta)$ for $\xi, \eta < \omega_1$ with $\xi \neq \eta$, be a subfamily of α consisting of non-empty sets. Set $\bar{\mu} = \{\text{cl}_X(V_{r_\xi, \gamma_\xi}) \mid \xi < \omega_1\}$. Then the family $\bar{\mu}$ has empty intersection. Indeed, suppose that some point $x = (x(\gamma))_{\gamma \in \Gamma}$ of X belongs to every element of $\bar{\mu}$. Since $\text{cl}_X(V_{r_\xi, \gamma_\xi}) \subseteq \{y \in X \mid y(\gamma_\xi) \geq r_\xi\}$ for every $\xi < \omega_1$, we get that $x(\gamma_\xi) \geq r_\xi$ for every $\xi < \omega_1$. From the countability of \mathbb{Q}_1 , we obtain that there exists $\xi_0 < \omega_1$ such that $|\{\xi < \omega_1 \mid r_\xi = r_{\xi_0}\}| = \aleph_1$. Hence $x(\gamma_\xi) \geq r_{\xi_0} > 0$ for every $\xi < \omega_1$ such that $r_\xi = r_{\xi_0}$. This implies that $|\{\gamma \mid x(\gamma) \neq 0\}| \geq \aleph_1 > \aleph_0$. Since $x \in \Sigma(\mathbb{I}, \Gamma)$, we get a contradiction. Therefore, $\bigcap \bar{\mu} = \emptyset$. Now, the compactness of X implies that there exists a finite subfamily $\bar{\mu}_0 = \{\text{cl}_X(V_{r_{\xi_i}, \gamma_{\xi_i}}) \mid i = 1, \dots, k\}$ (where $k \in \mathbb{N}$) of $\bar{\mu}$ with empty intersection. Then the family $\mu_0 = \{V_{r_{\xi_i}, \gamma_{\xi_i}} \mid i = 1, \dots, k\}$ is a finite subfamily of μ with empty intersection. Hence, α is a strongly point-countable family.

It is easy to see that α is an F-separating family. Now, we finish the proof as in ECs' case.

(\Leftarrow) It is completely analogous to the corresponding part of the proof for Eberlein compacts, however, Theorem 2.3 has to be used instead of Theorem 2.1.

The proof for uniform Eberlein compacts is the following:

(\Rightarrow) Let X be a UEC. Then X has a σ -boundedly point-finite, T_0 -separating family α of cozero-subsets (see Theorem 2.2). Let α^* be the family described in Lemma 2.14. Then α^* is an F-separating, σ -boundedly point-finite family. Thus, by Lemma 2.16, α^* is a σ -boundedly point-finite uniform almost subbase.

(\Leftarrow) It is completely analogous to the corresponding part of the proof for Eberlein compacts, however, Theorem 2.2 has to be used instead of Theorem 2.1.

The proof for n -uniform Eberlein compacts is completely analogous to that for uniform Eberlein compacts, however, in it the definition of n -uniform Eberlein compacts has to be used instead of Theorem 2.2. \square

Definition 3.5 ([16]) Let $n \in \omega$. A space X is called an *Eberlein space* (briefly, *E-space*) (respectively, *C-space*, *UE-space*, *n-UE-space*) if X has a σ -strongly point-finite (respectively, strongly point-countable, σ -boundedly point-finite, σ -($n+1$)-boundedly point-finite) almost subbase. A space X is said to be a *SE-space* if it has a strongly point-finite family α of clopen subsets such that $\alpha \cup \{X \setminus V \mid V \in \alpha\}$ is a subbase of X .

The assertions which are contained in the next theorem were announced in [16] with the exception of that about n -uniform Eberlein compactifications.

Theorem 3.6 ([16]) *Let $n \in \omega$. A space X has a compactification cX which is an Eberlein compact (respectively, Corson compact; uniform Eberlein compact; n -uniform Eberlein compact) iff it is an E-space (respectively, C-space; UE-space; n -UE-space).*

Proof. We start with the proof for Eberlein compactifications.

(\Rightarrow) Let X has a compactification cX which is an EC. We can think that X is a subspace of cX . By Theorem 3.4, cX has a σ -strongly point-finite almost subbase α' . Then Proposition 2.13 implies that $\alpha = \alpha' \cap X$ is an almost subbase of X . Obviously, α is a σ -strongly point-finite family. Hence, X is an E-space.

(\Leftarrow) Let X be an E-space. We will construct a compactification cX of X which is an Eberlein compact.

Let α be a σ -strongly point-finite almost subbase of X . For every $V \in \alpha$, we set $\mathbb{I}_V = \mathbb{I}$. Let $Z = \prod_{V \in \alpha} \mathbb{I}_V$. Let, for every $V \in \alpha$, $f_V : X \rightarrow I_V$ be the continuous function corresponding to the given U-representation $U(V)$ of V (see Lemma 2.6). Let $f = \Delta_{V \in \alpha} f_V : X \rightarrow Z$ be the diagonal of the mappings $(f_V)_{V \in \alpha}$. By Remark 2.8, f is a continuous injection. Let $cX = \text{cl}_Z(f(X))$ and let, for any $V \in \alpha$, $\pi_V : Z \rightarrow I_V$ be the projection. For every $V \in \alpha$, define a function $f'_V : cX \rightarrow I_V$ by the formula $f'_V = \pi_V|_{cX}$. Then, for every $V \in \alpha$ and every $n \in \mathbb{N}$, $f'_V(f(x)) = f_V(x)$ for every $x \in X$, $f(V) = f(X) \cap \text{coz}(f'_V)$ and $f(X \setminus U_{2n-1}(V)) = f(X) \cap (f'_V)^{-1}([0, \frac{1}{2^{n-1}}))$; hence, $f(V)$ and $f(X \setminus U_{2n-1}(V))$ are open subsets of $f(X)$. Thus, using Lemma 2.12, we get that f is an embedding of X into cX . Therefore, cX is a compactification of X . We will show that cX is an Eberlein compact.

Since, for any $V \in \alpha$, we have that $f(X) \cap \text{coz}(f'_V) = f(V)$, we obtain that $\text{coz}(f'_V) \subseteq \text{Ex}_{cX}(f(V))$. The family $\{f(V) \mid V \in \alpha\}$ is σ -strongly point-finite. Thus, by Lemma 3.3, the family $\{\text{Ex}_{cX}(f(V)) \mid V \in \alpha\}$ is σ -strongly point-finite.

Therefore, the family $\alpha' = \{\text{coz}(f'_V) \mid V \in \alpha\}$ is σ -point-finite. Adding to α' the family $\{(f'_V)^{-1}((r, 1]) \mid V \in \alpha, r \in \mathbb{Q}_1\}$, we get a T_0 -separating, σ -point-finite family of cozero-subsets of cX . Hence, using Rosenthal's Theorem 2.1, we obtain that cX is an Eberlein compact.

The proof for Corson (respectively, uniform Eberlein; n -uniform Eberlein) compactifications is completely analogous to that for Eberlein compactifications. However, in it we have to use Theorem 2.3 (respectively, Theorem 2.2; the definition of n -uniform Eberlein compacts) instead of Theorem 2.1. \square

Corollary 3.7 *Let $n \in \omega$. A space X is a subspace of an Eberlein compact (respectively, CC; UEC; n -UEC) iff it is an E -space (respectively, C -space; UE -space; n - UE -space).*

Proof. It follows from Theorem 3.6 and the fact that the closed subspaces of Eberlein compacts (respectively, CC; UEC; n -UEC) are Eberlein compacts (respectively, CC; UEC; n -UEC). \square

Corollary 3.8 ([33]) *Every metrizable space has a compactification which is a 0-uniform Eberlein compact.*

Proof. It follows from Theorem 3.6, the Bing Metrization Theorem and the fact that in metrizable spaces all open subsets are cozero-sets. \square

Remark 3.9 In [16], I mentioned that Theorem 3.6 implies the Arhangel'skii Theorem [5, Theorem 14] but I missed to note that Theorem 3.6 implies even a stronger result, namely that every metrizable space has a compactification which is a uniform Eberlein compact (the proof of both these results coincide with the proof of Corollary 3.8). I discovered this corollary a little bit later ([17]) and I intended to include it in the full version of the paper [16]. The writing of the full version was always postponed and in the meantime this assertion was noted by T. Banach and A. Leiderman [10] (and supplied with a short analytic proof). Finally, B. A. Pasynkov [33] obtained the result stated in Corollary 3.8, which is stronger than both previous assertions. In fact, B. A. Pasynkov proved in [33] a theorem which is even stronger than Corollary 3.8 (see [33, Theorem 4]).

Corollary 3.10 ([16]) *Every Baire E -space contains a dense G_δ metrizable subspace.*

Proof. It follows from theorems 3.6 and 2.10. \square

Fact 3.11 *A compact Hausdorff space X is a strong Eberlein compact iff it has a strongly point-finite family α consisting of clopen sets such that $\alpha \cup \{X \setminus V \mid V \in \alpha\}$ is a subbase of X .*

Proof. (\Rightarrow) Let X be a SEC. Then X has a T_0 -separating, point-finite family α of clopen sets. Since α consists of closed subsets of X and X is compact, we get that α is a strongly point-finite family. Using again the compactness of X , we obtain that the family $\alpha \cup \{X \setminus V \mid V \in \alpha\}$ is a subbase of X .

(\Leftarrow) Let α be a strongly point-finite family of clopen subsets of X such that $\alpha \cup \{X \setminus V \mid V \in \alpha\}$ is a subbase of X . Then, clearly, α is a T_0 -separating family. Therefore, X is a SEC. \square

Theorem 3.12 ([16]) *A space X has a compactification cX which is a strong Eberlein compact iff it is an SE-space.*

Proof. (\Rightarrow) Let X has a compactification cX which is a strong Eberlein compact. We can think that X is a subspace of cX . By Fact 3.11, cX has a strongly point-finite family α' consisting of clopen subsets of cX such that $\alpha' \cup \{cX \setminus V \mid V \in \alpha'\}$ is a subbase of cX . Then it is easy to see that $\alpha = \alpha' \cap X$ is a strongly point-finite family consisting of clopen subsets of X such that $\alpha \cup \{X \setminus V \mid V \in \alpha\}$ is a subbase of X . Hence, X is an SE-space.

(\Leftarrow) Let X be an SE-space and α be a strongly point-finite family of clopen subsets of X such that $\alpha \cup \{X \setminus V \mid V \in \alpha\}$ is a subbase of X . Setting, for every $V \in \alpha$ and every $n \in \mathbb{N}$, $U_n(V) = V$, we get, obviously, a U-representation $U(\alpha)$ of α which certify that α is an almost subbase of X . Let, for every $V \in \alpha$, $f_V : X \rightarrow \mathbb{D}$ be the characteristic function of V . Note that the function f_V corresponds to the U-representation $U(V)$ of V (see Lemma 2.6). Set $Y = \mathbb{D}^\alpha$ and let $f : X \rightarrow Y$ be the diagonal of the family $\{f_V \mid V \in \alpha\}$. Now, exactly as in the proof of Theorem 3.6, we show that $f : X \rightarrow Y$ is an embedding. Set $cX = \text{cl}_Y(f(X))$. We will show that cX is a strong Eberlein compact. As in the proof of Theorem 3.6, we define the functions $f'_V : cX \rightarrow \mathbb{D}$ for every $V \in \alpha$, and then, using Lemma 3.3, we obtain that the family $\alpha' = \{\text{coz}(f'_V) \mid V \in \alpha\}$ is strongly point-finite. Since, for every $V \in \alpha$, $\text{coz}(f'_V) = (f'_V)^{-1}(1)$, we get that α' consists of clopen subsets of cX . Also, it is clear that α' is a T_0 -separating family. Therefore, cX is a strong Eberlein compact. \square

Remark 3.13 In connection with the next corollary, let us note that, obviously, the closed subspaces of n -UECs are n -UECs (for every $n \in \omega$); also, the countable product of n -UECs is an n -UEC (for every $n \in \omega$). The proof of the last assertion is similar to the proof of Proposition 2 of [33]. Let us sketch it. Let $n \in \omega$ and $X = \prod \{X_i \mid i \in \mathbb{N}\}$, where, for every $i \in \mathbb{N}$, X_i is an n -UEC. Then, for every $i \in \mathbb{N}$, there exists a σ -($n+1$)-boundedly point-finite family γ_i of cozero subsets of X_i which T_0 -separates X_i . Let, for every $i \in \mathbb{N}$, $\pi_i : X \rightarrow X_i$ be the projection. Set, for every $i \in \mathbb{N}$, $\delta_i = \{\pi_i^{-1}(U) \mid U \in \gamma_i\}$ and $\delta = \bigcup \{\delta_i \mid i \in \mathbb{N}\}$. Then δ is σ -($n+1$)-boundedly point-finite family of cozero subsets of X which T_0 -separates X . Thus, X is an n -UEC.

Corollary 3.14 ([16]) *Let $n \in \omega$. Then:*

- (a) *The property of being an E-space (respectively, C-space, UE-space, n-UE-space, SE-space) is hereditary and additive;*
- (b) *The property of being E-space (respectively, C-space, UE-space, n-UE-space) is \aleph_0 -multiplicative;*
- (c) *If X is a C-space (respectively, E-space, UE-space, n-UE-space, SE-space), then X is a Fréchet-Urysohn space;*
- (d) *If X is an E-space, then $c(X) = w(X) = d(X)$, $c(X) \leq |X| \leq (c(X))^{\aleph_0}$, $\aleph_0 \leq \chi(X) \leq c(X)$ and the bounds given in the last two inequalities are the best possible;*
- (e) *The following conditions are equivalent for a space X : (i) X is separable metrizable; (ii) X is a 0-UE-space with $c(X) = \aleph_0$; (iii) X is a UE-space with $c(X) = \aleph_0$; (iv) X is an E-space with $c(X) = \aleph_0$.*

Proof. (a) The fact that the corresponding property is hereditary is obvious. Let J be a set, $\{X_j \mid j \in J\}$ be a disjoint family of E-spaces and $X = \bigoplus \{X_j \mid j \in J\}$. Let, for every $j \in J$, α_j be a σ -strongly point-finite almost subbase of X_j . Then $\alpha = \{X_j \mid j \in J\} \cup \bigcup \{\alpha_j \mid j \in J\}$ is a σ -strongly point-finite almost subbase of X . Hence, X is an E-space. The proof for the other three cases is analogous.

(b) It follows from Theorem 3.6 and the fact that the class of ECs (respectively, CCs, UECs, n -UECs) is closed under countable products (see [29, Proposition 3.3], [44, Theorem 3.6] and Remark 3.13).

(c) It follows from Theorem 3.6, the fact that Corson compacts are Fréchet-Urysohn spaces (see, e.g., [21, 3.10.D]) and [21, 2.1.H(b)].

(d) By Theorem 3.6, X has a compactification Y which is an EC. Then $c(Y) = w(Y)$ (see [44, Theorem 3.1]). Using [21, 2.7.9(d)], we get that $w(X) \leq w(Y) = c(Y) = c(X) \leq w(X)$. Thus $c(X) = w(X)$. Since $c(X) \leq d(X) \leq w(X)$, we get that $d(X) = w(X) = c(X)$. Further, by [44, Theorem 3.1], $c(Y) \leq |Y| \leq (c(Y))^{\aleph_0}$. Then $c(X) \leq |X| \leq |Y| \leq (c(Y))^{\aleph_0} = (c(X))^{\aleph_0}$. Finally, we have that $\aleph_0 \leq \chi(X) \leq w(X) = c(X)$. The rest follows from the remark after [44, Theorem 3.1].

(e) It follows from (d) and Corollary 3.8. □

Corollary 3.15 *Let $n \in \omega$ and X be a locally compact Hausdorff space. Then:*

- (a) ([16]) *the Alexandroff compactification αX of X is an EC (respectively, CC, UEC, SEC) iff X is an E-space (respectively, C-space, UE-space, SE-space);*
- (b) *if, in addition, X is a paracompact space then αX is an n -UEC iff X is an n -UE-space; thus, if X is metrizable, then αX is a 0-UEC;*
- (c) ([16]) *if Y is a perfect image of X and X is an E-space (respectively, C-space, UE-space, SE-space) then Y is an E-space (respectively, C-space, UE-space, SE-space).*

Proof. (a) The necessity follows from theorems 3.6 and 3.12. For the sufficiency, let X be an E-space. Then, by Theorem 3.6, X has a compactification cX which is an EC. Since αX is a continuous image of cX , Theorem 2.4 implies that αX is an EC. The proof for the other three cases is analogous.

(b) The necessity is clear. Let us prove the sufficiency. Let X be a paracompact locally compact Hausdorff n -UE-space. By the Morita Theorem (see [21, Theorem 5.1.27]), we have that $X = \bigoplus \{X_j \mid j \in J\}$, where J is some set and, for every $j \in J$, X_j is a Lindelöf space. Then, for every $j \in J$, every open F_σ -subset of X_j is a union of countably many compact subsets of X_j , and thus it is a cozero-subset of αX . Therefore, if, for every $j \in J$, γ_j is a σ -($n+1$)-boundedly point-finite family of cozero-subsets of X_j that T_0 -separates X_j , then $\gamma = \{X_j \mid j \in J\} \cup \bigcup \{\gamma_j \mid j \in J\}$ is a σ -($n+1$)-boundedly point-finite family of cozero-subsets of αX that T_0 -separates αX . Hence, αX is an n -UEC.

Since every metrizable space X is a 0-UE-space (see Corollary 3.8), the above result implies that if, in addition, X is locally compact then αX is a 0-UEC.

(c) Let $f : X \longrightarrow Y$ be a perfect surjection. Then f can be extended to a continuous map $f^\alpha : \alpha X \longrightarrow \alpha Y$ and all follows from (a) and Theorem 2.4. \square

Remark 3.16 (a) Note that in the proof of Corollary 3.15(a) we used Theorem 2.4 which is a very deep result. If we suppose in Corollary 3.15(a) that X is, in addition, a paracompact space, then the obtained weaker assertion for E-spaces, C-spaces and UE-spaces can be proved as Corollary 3.15(b).

(b) If the continuous image of an n -UEC, where $n \in \omega$, were an n -UEC, then we could remove the requirement of paracompactness in Corollary 3.15(b) arguing as in the proof of Corollary 3.15(a). But, *in general, the continuous image of an n -UEC, where $n \in \omega$, is not an n -UEC*. Indeed, let, for every cardinal number $\tau \geq \aleph_0$, D_τ denote the discrete space of cardinality τ . Then αD_τ is a 0-UEC. Hence, by Remark 3.13, $X_\tau = (\alpha D_\tau)^{\aleph_0}$ is a 0-UEC, as well as every closed subset of X_τ . By [11, Lemma 1.2], for every UEC Y there exists a $\tau \geq \aleph_0$ and a closed subset F of X_τ such that Y is a continuous image of F . Thus, *every UEC is a continuous image of a 0-UEC*. Since there exist UECs which are not 0-UECs (see [33]), we obtain that the continuous image of a 0-UEC is not, in general, a 0-UEC.

Corollary 3.17 ([29]) *If X is locally compact metrizable or if X is a disjoint sum of ECs then the Alexandroff compactification αX of X is an EC.*

Proof. It follows from corollaries 3.8, 3.14(a) and Remark 3.16(a). \square

Corollary 3.18 *Let $n \in \omega$. If X is a disjoint sum of n -UECs then the Alexandroff compactification αX of X is an n -UEC.*

Proof. It follows from [21, Theorem 5.1.30] and corollaries 3.14(a), 3.15(b). \square

Definition 3.19 ([16]) A space X is called σ -strongly metacompact (respectively, strongly metalindelöf) if every open cover of X has a σ -strongly point-finite (respectively, strongly point-countable) open refinement.

In what follows, if \mathcal{A} is a family of subsets of a space X then we shall denote by $\overline{\mathcal{A}}$ the family $\{\text{cl}_X(A) \mid A \in \mathcal{A}\}$.

Lemma 3.20 ([17]) *Let X be a locally compact Hausdorff space. Then:*

- (a) X is (hereditarily) strongly metalindelöf iff it is (hereditarily) metalindelöf;
- (b) X is (hereditarily) σ -strongly metacompact iff it is (hereditarily) σ -metacompact.

Proof. (a) The necessity is obvious. For the sufficiency, let us first regard the case when X is a metalindelöf locally compact Hausdorff space. Let \mathcal{U} be an open cover of X . Then \mathcal{U} has an open refinement \mathcal{U}' whose elements have compact closures. Let \mathcal{V} be an open point-countable refinement of \mathcal{U}' . Then, by [25, Theorem 1.1 and Footnote 1], there exists an open cover $\mathcal{W} = \{W_V \mid V \in \mathcal{V}\}$ of X such that $\text{cl}_X(W_V) \subseteq V$ for every $V \in \mathcal{V}$. Let $\mu \subseteq \mathcal{W}$ and $|\mu| = \aleph_1$. Then $\bigcap \overline{\mu} = \emptyset$ because $\overline{\mathcal{W}}$ is a shrinking of \mathcal{V} and \mathcal{V} is point-countable. Since the elements of $\overline{\mu}$ are compact sets, there exists a finite subfamily μ_0 of μ such that $\bigcap \overline{\mu_0} = \emptyset$. Then $\bigcap \mu_0 = \emptyset$. Hence, the cover \mathcal{W} is strongly point-countable. This implies that X is strongly metalindelöf.

Let now X be a hereditarily metalindelöf locally compact Hausdorff space. Then, clearly, by the above paragraph, every open subspace of X is strongly metalindelöf. This implies easily that X is hereditarily strongly metalindelöf.

(b) The proof is similar to that one of (a). □

Remark 3.21 A space X is called *strongly metacompact* ([34]) (note that the name “b-paracompact space” is used in [34]) if every open cover of X has a strongly point-finite open refinement. In [34], it is proved that if X is a locally compact Hausdorff space, then X is strongly metacompact iff it is metacompact (as it is written there, this is a result of M. Patashnik). Obviously, this implies that if X is a locally compact Hausdorff space, then X is hereditarily strongly metacompact iff it is hereditarily metacompact.

Let us also note that in [40] the results stated in Lemma 3.20 were mentioned using [17] as a prime source.

Using Lemma 3.20 and Yakovlev’s Theorem ([46, Corollary 2]), we obtain:

Corollary 3.22 ([16]) *Every Eberlein space (respectively, C -space) is hereditarily σ -strongly metacompact (respectively, hereditarily strongly metalindelöf).*

Remark 3.23 Recall that, by the Gruenhage Theorem [24, Theorem 2.2], a compact Hausdorff space is an Eberlein compact iff X^2 is hereditarily σ -metacompact. In 1991, I asked S. Popvassilev (who was my student at that time) *whether a Tychonoff space X is an E-space iff X^2 is hereditarily σ -strongly metacompact*; in his Master Thesis (University of Sofia, 1992) he observed that *the space $X = C_p(\mathbb{I})$ is a non-E-space* (because $c(X) = \aleph_0 < 2^{\aleph_0} = |\mathbb{I}| = w(X)$ by [8, Theorems 0.3.7 and I.1.1], and this contradicts Corollary 3.14(d)) *but X^2 is hereditarily σ -strongly metacompact* (indeed, X^2 is hereditarily Lindelöf by the Zenor-Velichko Theorem (see [8, Theorem II.5.10]); thus, by [21, Corollary 5.3.11], X^2 is hereditarily strongly paracompact and, hence, X^2 is hereditarily (σ) -strongly metacompact).

S. A. Peregudov [34, Lemma 3] proved that in every space X the cardinality of every strongly point-finite family of open sets is less or equal to $c(X)$. This result implies that if X is σ -strongly metacompact then $l(X) \leq c(X)$ (see [40]) (here, as usual, $l(X)$ is the Linelöf number of X , i.e., the smallest cardinal number τ such that every open cover of X has an open refinement of cardinality $\leq \tau$). Thus, using Corollary 3.22, we obtain the following fact:

Corollary 3.24 *If X is an E-space, then $l(X) \leq c(X)$.*

Let us note that S. A. Peregudov [34, Theorem 2] proved that if X is a strongly metacompact space then $l(X) \leq c(X)$.

Corollary 3.25 *If X is an E-space, then $hl(X) = c(X) = hc(X)$.*

Proof. By Corollary 3.14(d), we have that $c(X) = w(X)$. Since, obviously, $c(X) \leq hc(X) \leq w(X)$, we get that $c(X) = hc(X)$. Let $A \subseteq X$. Then, by Corollaries 3.14(a) and 3.24, $l(A) \leq c(A) \leq hc(X) = c(X)$. Therefore, $hl(X) \leq c(X)$. Since $hl(X) \geq c(X)$ (see, e.g., [21, 3.12.7(e)]), we get that $hl(X) = c(X)$. \square

Remark 3.26 By the Arhangel'skiĭ Theorem [5, Theorem 14], every metrizable space is an E-space. On the other hand, as it is noted in [7], “not every completely regular Moore space has an Eberlein compactification, since there are separable non-metrizable Moore spaces”. Such a space is, for example, the Niemytzki plane (see, e.g., [14]). By the Yakovlev Theorem ([46, Corollary 2]), every Eberlein compact is hereditarily σ -metacompact. Hence, it is natural to ask whether a Tychonoff (or normal) metacompact Moore space is an E-space. Since a regular space has a uniform base (in the sense of P. S. Alexandrov [1]) iff it is a metacompact Moore space ([1, Corollary and Theorem III], [27, Theorem 4]), this question is equivalent to the following one: *is every Tychonoff (or normal T_1) space with a uniform base an E-space?* (Let us recall that, by [1], a space is metrizable iff it is a collectionwise normal T_1 -space with a uniform base.)

The Niemytzki plane is not a space with a uniform base because it is not a metacompact space (see [21, Exercise 5.3.B(a)]). Let us show that the Pixley-Roy

example X of a Tychonoff non-separable Moore metacompact space with $c(X) = \aleph_0$ (see [36]) *is a Tychonoff non-E-space with a uniform base*. Indeed, we have that:

- (a) X is a space with a uniform base,
 - (b) X is a p-space (in the sense of [6]) (because each completely regular Moore space is a p-space ([4] or [9, V.226])),
 - (c) as a space with a uniform base, X has a point-countable base (see [1, Prop. I]).
- Supposing that X is a Lindelöf space, we obtain that X is metrizable (because each paracompact p-space with a point-countable base is metrizable (see [37] or [9, V.229])), and, thus, $c(X) = d(X)$, a contradiction. Hence, X is a non-Lindelöf space, i.e., $l(X) > \aleph_0$. Since $c(X) = \aleph_0$, Corollary 3.24 implies that X is a non-E-space.

Starting with the Przymusiński-Tall example of a normal non-separable Moore metacompact T_1 -space X with $c(X) = \aleph_0$, constructed under $MA + \neg CH$ in [41], we obtain, as above, that X is a non-E-space. Therefore, *under $MA + \neg CH$, there exists a normal Hausdorff non-E-space with a uniform base*.

P. J. Nyikos [32] proved that the Product Measure Extension Axiom (PMEA, for short) implies that normal Moore T_1 -spaces are metrizable. Thus, *under PMEA, every normal T_1 -space with a uniform base is metrizable and, hence, it is an E-space*.

4 On spaces co-absolute with Eberlein compacts and Ponomarev's solution of Birkhoff's Problem 72

The fact that two Boolean algebras A and B are isomorphic will be expressed by " $A \cong B$ ".

Notation 4.1 We shall denote by:

- \mathcal{M} the class of all metrizable spaces,
- \mathcal{E} the class of all Eberlein compacts.

If \mathcal{K} is a class of topological spaces, we will set

$$\mathcal{BK} = \{A \in |\mathbf{Bool}| \mid (\exists X \in \mathcal{K})(A \cong RO(X))\}.$$

Recall that a subset B of a Boolean algebra A is said to be σ -disjointed (respectively, *dense*) if $B = \bigcup \{B_n \mid n \in \mathbb{N}\}$, where for every $n \in \mathbb{N}$ and for every two different elements a, b of B_n we have $a \wedge b = 0$ (respectively, if for any $a \in A \setminus \{0\}$ there exists $b \in B \setminus \{0\}$ such that $b \leq a$).

The Problem 72 of G. Birkhoff [13] is the following: characterize internally the elements of the class \mathcal{BM} . It was solved by V. I. Ponomarev [38]. He proved the following beautiful theorem: if A is a complete Boolean algebra then $A \in \mathcal{BM}$ iff it has a σ -disjointed dense subset B . The proof of this theorem is difficult. We

will obtain a shorter proof of it on the base of some results from [5, 31, 18] which appeared many years after the publication of Ponomarev's paper [38].

We will need a lemma from [15]:

Lemma 4.2 *Let X be a dense subspace of a topological space Y . Then the Boolean algebras $RC(X)$ and $RC(Y)$ are isomorphic.*

If X is a set and $\gamma = \bigcup\{\gamma_n \mid n \in \mathbb{N}\}$, where, for every $n \in \mathbb{N}$, γ_n is a disjoint family of subsets of X , then we will say that γ is a σ -disjoint family. (Obviously, “ σ -disjoint family” = “ σ -1-boundedly point-finite family”.)

The new proof of the Ponomarev Theorem as well as the characterization of the class of spaces which are co-absolute with (zero-dimensional) Eberlein compacts will follow from the next theorem.

Theorem 4.3 *A complete Boolean algebra A is isomorphic to an algebra of the form $RC(X)$, where X is a (zero-dimensional) Eberlein compact, iff A has a σ -disjointed dense subset.*

Proof. (\Rightarrow) Let A be a Boolean algebra which is isomorphic to $RC(X)$, where X is an Eberlein compact. By Corollary 2.11, there exists a metrizable dense subset Y of X . Hence, by Lemma 4.2, A is isomorphic to $RC(Y)$. The space Y has a σ -discrete base $\mathcal{B} = \bigcup\{\mathcal{B}_i \mid i \in \mathbb{N}^+\}$, where \mathcal{B}_i is a discrete family for every $i \in \mathbb{N}^+$. Set, for every $i \in \mathbb{N}^+$, $\mathcal{B}'_i = \{\text{cl}(U) \mid U \in \mathcal{B}_i\}$, and let $\mathcal{B}' = \bigcup\{\mathcal{B}'_i \mid i \in \mathbb{N}^+\}$. Then, obviously, \mathcal{B}' is a σ -disjointed dense subset of $RC(Y)$. Hence, A has a σ -disjointed dense subset.

(\Leftarrow) Let A be a complete Boolean algebra having a σ -disjointed dense subset B_0 . Let B be the Boolean subalgebra of A generated by B_0 . Then A is a minimal completion of B . Set $X = S(B)$. Then X is a zero-dimensional compact Hausdorff space and there exists an isomorphism $\varphi : B \rightarrow CO(X)$. We will show that $\mathcal{B} = \varphi(B_0)$ is a σ -disjoint almost subbase of X . For every $V \in \mathcal{B}$ and every $n \in \mathbb{N}^+$, set $U_n(V) = V$. Then $\{U_n(V) \mid i \in \mathbb{N}^+\}$ is a U-representation of V . Hence, it is enough to show that the family $\mathcal{B}' = \mathcal{B} \cup \{X \setminus V \mid V \in \mathcal{B}\}$ is a subbase of X . Obviously, $\mathcal{B}' = \varphi(B_0 \cup B_0^*)$, where $B_0^* = \{b^* \mid b \in B_0\}$. Since the set of all finite joins of all finite meets of the elements of the subset $B_0 \cup B_0^*$ of A coincides with B (see [28, Proposition 4.4]), we get that the family of all finite unions of the finite intersections of the elements of the family \mathcal{B}' coincides with $CO(X)$. The family $CO(X)$ is a base of X ; hence, the family of all finite intersections of the elements of \mathcal{B}' is a base of X , i.e., \mathcal{B}' is a subbase of X . Therefore, \mathcal{B} is an almost subbase of X . Since \mathcal{B} is, obviously, a σ -disjoint family, we get, by Theorem 2.17 (or by Theorem 3.4), that X is an Eberlein compact. Now, $RC(X)$ is a minimal completion of $CO(X)$; thus $RC(X)$ and A are isomorphic Boolean algebras. \square

Remark 4.4 Note that the zero-dimensional Eberlein compact X constructed in the proof of the sufficiency of the preceding theorem is even a 0-UEC (by Theorem 3.4).

Proposition 4.5 $\mathcal{BM} = \mathcal{BE}$.

Proof. By the Arhangel'skii Theorem [5, Theorem 14], every metric space can be densely embedded in an Eberlein compact. Conversely, by Corollary 2.11, every Eberlein compact contains a dense metrizable subspace. Applying Lemma 4.2, we conclude that $\mathcal{BM} = \mathcal{BE}$. \square

Combining Theorem 4.3 with Proposition 4.5, we obtain the Ponomarev Theorem [38] giving a solution of Birkhoff's Problem 72 [13].

Corollary 4.6 (V. I. Ponomarev [38]) *A complete Boolean algebra A is isomorphic to an algebra of the form $RC(X)$, where X is a metrizable space, iff A has a σ -disjointed dense subset.*

Recall that if X is a regular space then a space EX is called an *absolute* of X iff there exists a perfect irreducible map $\pi_X : EX \rightarrow X$ and every perfect irreducible preimage of EX is homeomorphic to EX (see, e.g., [39]). Two regular spaces are said to be *co-absolute* if their absolutes are homeomorphic. It is well-known that: a) the absolute is unique up to homeomorphism; b) a space Y is an absolute of a regular space X iff Y is an extremally disconnected Tychonoff space for which there exists a perfect irreducible map $\pi_X : Y \rightarrow X$; c) if X is a compact Hausdorff space then $EX = S(RC(X))$ (see, e.g., [45]).

Theorem 4.7 *Let X be a compact Hausdorff space. Then the following conditions are equivalent:*

- (a) *X is co-absolute with an Eberlein compact;*
- (b) *X has a σ -disjoint π -base;*
- (c) *X is co-absolute with a zero-dimensional 0-UEC;*
- (d) *X is co-absolute with a zero-dimensional Eberlein compact.*

Proof. (a) \Rightarrow (b) Let Y be an Eberlein compact which is co-absolute with X . Then $RC(Y) \cong RC(X)$. By Theorem 4.3, we get that the Boolean algebra $RO(X)$ (which is isomorphic to the Boolean algebra $RC(X)$) has a σ -disjointed dense subset \mathcal{A} . Then, obviously, \mathcal{A} is a σ -disjoint π -base of X .

(b) \Rightarrow (c) Let \mathcal{A} be a σ -disjoint π -base of X . Set $\mathcal{A}' = \{\text{int}(\text{cl}(U)) \mid U \in \mathcal{A}\}$. Then, obviously, \mathcal{A}' is a σ -disjointed dense subset of the Boolean algebra $RO(X)$. Since $RO(X) \cong RC(X)$, Remark 4.4 implies that there exists a zero-dimensional 0-UEC Y with $RC(Y) \cong RC(X)$. Thus X and Y are co-absolute spaces.

(c) \Rightarrow (d) and (d) \Rightarrow (a) Both implications are obvious. \square

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